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Reconstruction of a convolution kernel in a semilinear parabolic problem based on a global measurement

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Abstract

A semilinear parabolic problem of second order with an unknown time-convolution kernel is considered. The missing kernel is recovered from an additional integral measurement. The existence, uniqueness and regularity of a weak solution is addressed. We design a numerical algorithm based on Rothe's method, derive a priori estimates and prove convergence of iterates towards the exact solution.

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1. Introduction

The general nature of an inverse problem (IP) is to deduce a cause from an effect. IPs typically lead to mathematical models that are ill-posed in the sense of Hadamard – see [1]. Moreover, ill-posed problems frequently turn out to be numerically unstable (sensitive to small errors in the known data), in that small changes in the known data may lead to arbitrarily large changes in the response. Many IPs do not have a solution in the strict classical sense, or if there is a solution, it might not be unique or might not depend continuously on the data. To obtain global in time existence and uniqueness of a solution is in general a very difficult part of the problem. The second important component of the task is to describe a constructive way how to find the solution. The usual algorithms start with parametrization of the problem and they make use of continuous dependence of a parametrized solution on the parameter. An error/cost functional is constructed and minimized in suitable function spaces linked to the setting under consideration. The bottleneck of this approach is convexity of the functional, caused by ill-posedness of the IP. In most cases the missing convexity is remediated by an appropriate regularization cf. e.g. [2, 3, 4]. The Tikhonov-regularization is based on adding a suitable term to the functional in order to guarantee its convexity, ensuring the existence of a unique solution to the minimization problem. This later problem can be solved numerically by adequate approximation techniques, such as the steepest descend, Ritz or Newton or Levenberg-Marquardt method, see e.g. [5, 6].

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In this paper, we are interested in determining of the unknown couple $\langle u, K \rangle$ obeying the following semilinear parabolic problem

$$\begin{cases} \partial_t u(\mathbf{x}, t) - \Delta u(\mathbf{x}, t) + K(t)h(\mathbf{x}, t) + (K * u(\mathbf{x}))(t) = f(\mathbf{x}, t, u(\mathbf{x}, t), \nabla u(\mathbf{x}, t)), & \text{in } \Omega \times I, \\ -\nabla u(\mathbf{x}, t) \cdot \nu = g(\mathbf{x}, t), & \text{on } \Gamma \times I, \\ u(\mathbf{x}, 0) = u_0(\mathbf{x}), & \text{in } \Omega, \end{cases} \quad (1)$$

where Ω is a Lipschitz domain (cf. [7]) in \mathbb{R}^N , $N \geq 1$, with $\partial\Omega = \Gamma$ and $I = [0, T]$, $T > 0$ in the time frame. By $K * u$ we denote the usual convolution in time, namely $(K * u(\mathbf{x}))(t) = \int_0^t K(t-s)u(\mathbf{x}, s) ds$. The missing time-convolution kernel $K = K(t)$ will be recovered from the following integral-type measurement

$$\int_{\Omega} u(\mathbf{x}, t) d\mathbf{x} = m(t), \quad t \in [0, T]. \quad (2)$$

The integral type over-determination in IPs combined with evolutionary PDEs has been studied in several papers, e.g. [8, 9, 10] and the references therein.

Such type of integro-differential problems arise for example elastoplasticity (cf. [11]) or in the theory of reactive contaminant transport. In [12] one considers the following differential equation

$$\partial_t C + \nabla \cdot (\mathbf{V}C) - \Delta C = \frac{-\rho_b}{n} \partial_t S$$

for the aqueous concentration C and sorbed concentration per unit mass of solid S with mass transformation rate in first order kinetics form of

$$\partial_t S = K_r(K_d C - S)$$

with desorption rate K_r and equilibrium distribution coefficient K_d . This is indeed a problem of type (1) for $u = C$ with $K(t) = -\frac{\rho_b}{n} K_r^2 K_d e^{-K_r t}$, $h(t) = -\frac{S_0}{K_r K_d}$ and $f(x, \mathbf{r}) = \frac{-\rho_b}{n} K_r K_d x - \mathbf{V} \cdot \mathbf{r}$.

Identification of missing memory kernels in evolutionary PDEs is relatively new in IPs. We are aware of only a few papers dealing with this topics, namely [13, 14, 15, 16, 17]. In [14] a global in time existence and uniqueness result for an inverse problem arising in the theory of heat conduction for materials with memory has been studied. The reference [17] derives some local and global in time existence results for the recovery of memory kernels. There is no description of constructive algorithms how to find a solution.

The main goal of this paper is to design a productive numerical scheme describing the way of retrieving the couple $\langle u, K \rangle$. This is achieved not by minimization of a cost functional (which is typical for IPs) but on the time discretization based on Rothe's method [18, 19]. First, we start with derivation of a suitable variational formulation. Section 2 is devoted to the study of regularity of a weak solution, and the uniqueness is addressed in Theorem 1. Section 3 deals with a time discretization, where (based on backward Euler scheme) the continuous problem is approximated by a sequence of steady state settings at each point of a time partitioning. Stability analysis of approximates is performed in appropriate function spaces and convergence (based on compactness argument) is established in Theorem 2.

Notations. Denote by (\cdot, \cdot) the standard inner product of $L^2(\Omega)$ and $\|\cdot\|$ its induced norm. When working at the boundary Γ we use a similar notation, namely $(\cdot, \cdot)_{\Gamma}$, $L^2(\Gamma)$ and $\|\cdot\|_{\Gamma}$. By $C([0, T], X)$ we denote the set of abstract functions $w : [0, T] \rightarrow X$ equipped with the usual norm $\max_{t \in [0, T]} \|\cdot\|_X$ and $L^p((0, T), X)$ is furnished with the norm

$$\left(\int_0^T \|\cdot\|_X^p dt \right)^{\frac{1}{p}} \text{ with } p > 1, \text{ cf. [20]. The symbol } X^* \text{ stands for the dual space to } X.$$

We take a test function $\phi \in H^1(\Omega)$, and derive from (1) after integration over Ω that

$$(\partial_t u, \phi) - (\Delta u, \phi) + K(h, \phi) + (K * u, \phi) = (f(u, \nabla u), \phi). \quad (3)$$

Make use of Green's first identity to obtain

$$(\partial_t u, \phi) + (\nabla u, \nabla \phi) + (g, \phi)_{\Gamma} + K(h, \phi) + (K * u, \phi) = (f(u, \nabla u), \phi), \quad (P)$$

If we set $\phi = 1$ in (P) we obtain together with the measurement $(u(t), 1) = m(t)$ that

$$m' + (g, 1)_\Gamma + K(h, 1) + K * m = (f(u, \nabla u), 1). \quad (\text{MP})$$

The relations (P) and (MP) represent the variational formulation of (1) and (2).

Finally, as is usual in papers of this sort, C, ε and C_ε will denote generic positive constants depending only on a priori known quantities, where ε is small and $C_\varepsilon = C(\varepsilon^{-1})$ is large.

2. Stability analysis of a solution, uniqueness

First, we start with a study of natural regularity of a solution $\langle u, K \rangle$. This helps us to choose appropriate function spaces for the variational framework. Uniqueness of a solution is addressed at the end of this section.

Proposition 2.1. *Let f be bounded, i.e. $|f| \leq C$. Moreover assume that $u_0 \in L^2(\Omega)$, $g \in C([0, T], L^2(\Gamma))$, $h \in C([0, T], L^2(\Omega))$, $\min_{t \in [0, T]} |(h(t), 1)| \geq \omega > 0$ and $m \in C^1([0, T])$. If $\langle u, K \rangle$ is a solution of (1) and (2), then K is bounded on $[0, T]$, i.e.*

$$\max_{t \in [0, T]} |K(t)| \leq C.$$

Proof. Take any $t \in [0, T]$. From (MP) it follows that

$$|K(t)(h(t), 1)| \leq |(f(u(t), \nabla u(t)), 1)| + |(K * m)(t)| + |m'(t)| + |(g(t), 1)_\Gamma|.$$

Involving the assumptions we see that

$$\omega |K(t)| \leq |(h(t), 1)| |K(t)| \leq C + |(K * m)(t)| \leq C + C \int_0^t |K(s)| \, ds.$$

We conclude the proof by Grönwall's argument, cf. [21]. \square

Proposition 2.2. *Let the conditions of Proposition 2.1 be satisfied. If $\langle u, K \rangle$ is a solution of (1) and (2), then there exists $C > 0$ such that*

$$(i) \max_{t \in [0, T]} \|u(t)\|^2 + \int_0^t \|\nabla u(\xi)\|^2 \, d\xi \leq C$$

$$(ii) \int_0^t \|\partial_t u\|_{(H^1(\Omega))^*}^2 \, d\xi \leq C.$$

Proof. (i) If we set $\phi = u$ in (P) and integrate in time over $(0, t)$ we obtain

$$\int_0^t (\partial_t u, u) \, d\xi + \int_0^t (\nabla u, \nabla u) \, d\xi + \int_0^t (g, u)_\Gamma \, d\xi + \int_0^t K(h, u) \, d\xi + \int_0^t (K * u, u) \, d\xi = \int_0^t (f(u, \nabla u), u) \, d\xi. \quad (4)$$

The first two terms can be rewritten as

$$\int_0^t (\partial_t u, u) \, d\xi = \frac{1}{2} \|u(t)\|^2 - \frac{1}{2} \|u_0\|^2, \quad \int_0^t (\nabla u, \nabla u) \, d\xi = \int_0^t \|\nabla u(\xi)\|^2 \, d\xi.$$

For the third one we get

$$\left| \int_0^t (g, u)_\Gamma \, d\xi \right| \leq \int_0^t \|g\|_\Gamma \|u\|_\Gamma \, d\xi \leq C \int_0^t \|g\|_\Gamma \|u\|_{H^1(\Omega)} \, d\xi \leq C_\varepsilon \int_0^t \|g\|_\Gamma^2 + \varepsilon \int_0^t \|u\|_{H^1(\Omega)}^2 \, d\xi$$

by Cauchy's inequality, the trace theorem and Young's inequality. The fourth term is easily bounded by

$$\left| \int_0^t K(h, u) \, d\xi \right| \leq \int_0^t |K| \|h\| \|u\| \, d\xi \leq C \int_0^t \|h\|^2 \, d\xi + C \int_0^t \|u\|^2 \, d\xi,$$

as K is bounded, see Proposition 2.1. It holds

$$\|(K * u)(t)\|^2 = \int_{\Omega} \left(\int_0^t K(t-s)u(\mathbf{x}, s) \, ds \right)^2 \, d\mathbf{x} \leq \int_{\Omega} \int_0^t K^2(t-s) \int_0^t u^2(\mathbf{x}, s) \, ds \, d\mathbf{x} \leq C \int_0^t \|u(s)\|^2 \, ds. \quad (5)$$

The last term in the left-hand side of (4) is

$$\left| \int_0^t (K * u, u) \, d\xi \right| \leq \int_0^t \|K * u\| \|u\| \, d\xi \leq \frac{1}{2} \int_0^t \|K * u\|^2 \, d\xi + \frac{1}{2} \int_0^t \|u\|^2 \, d\xi \leq C \int_0^t \|u\|^2 \, d\xi.$$

The right-hand side of (4) can be estimated as follows

$$\left| \int_0^t (f(u, \nabla u), u) \, d\xi \right| \leq \int_0^t \|f(u, \nabla u)\| \|u\| \, d\xi \leq \frac{1}{2} \int_0^t \|f(u, \nabla u)\|^2 \, d\xi + \frac{1}{2} \int_0^t \|u\|^2 \, d\xi \leq C + \frac{1}{2} \int_0^t \|u\|^2 \, d\xi,$$

as f is bounded.

Putting all things together, fixing a sufficiently small $\varepsilon > 0$ and taking into account $\|u\|_{H^1(\Omega)}^2 = \|u\|^2 + \|\nabla u\|^2$ we obtain

$$\|u(t)\|^2 + \int_0^t \|\nabla u(\xi)\|^2 \, d\xi \leq C + C \int_0^t \|u\|^2 \, d\xi,$$

which is valid for any $t \in [0, T]$. An application of Grönwall's lemma concludes the proof.

(ii) Starting from (P) and using the Cauchy inequality, Lemma 2.1, (5), trace theorem and Lemma 2.2(i) we successively deduce that

$$\begin{aligned} |(\partial_t u, \phi)| &= |(f(u, \nabla u), \phi) - (\nabla u, \nabla \phi) - (g, \phi)_{\Gamma} - K(h, \phi) + (K * u, \phi)| \\ &\leq C \left(\|\phi\| + \|\nabla u\| \|\nabla \phi\| + \|\phi\|_{\Gamma} + \sqrt{\int_0^t \|u\|^2} \|\phi\| \right) \\ &\leq C \left(\|\nabla u\| \|\nabla \phi\| + \|\phi\|_{H^1(\Omega)} \right). \end{aligned}$$

Thus $(\partial_t u, \phi)$ can be seen as a linear functional on $H^1(\Omega)$ and we may write

$$\|\partial_t u\|_{(H^1(\Omega))^*} = \sup_{\|\phi\|_{H^1(\Omega)} \leq 1} |(\partial_t u, \phi)| \leq C(1 + \|\nabla u\|),$$

which implies that

$$\int_0^T \|\partial_t u\|_{(H^1(\Omega))^*}^2 \, d\xi \leq C + C \int_0^T \|\nabla u\|^2 \, d\xi \leq C.$$

□

Proposition 2.3. *Let the conditions of Proposition 2.1 be satisfied and moreover $g \in C^1([0, T], L^2(\Gamma))$ and $u_0 \in H^1(\Omega)$. If $\langle u, K \rangle$ is a solution of (1) and (2), then there exists $C > 0$ such that*

$$\max_{t \in [0, T]} \|\nabla u(t)\|^2 + \int_0^T \|\partial_t u(\xi)\|^2 \, d\xi \leq C.$$

Proof. If we set $\phi = \partial_t u$ in (P) and integrate in time we obtain

$$\int_0^t (\partial_t u, \partial_t u) \, d\xi + \int_0^t (\nabla u, \nabla \partial_t u) \, d\xi + \int_0^t (g, \partial_t u)_{\Gamma} \, d\xi + \int_0^t K(h, \partial_t u) \, d\xi + \int_0^t (K * u, \partial_t u) \, d\xi = \int_0^t (f(u, \nabla u), \partial_t u) \, d\xi. \quad (6)$$

The first two terms can be rewritten as

$$\int_0^t (\partial_t u, \partial_t u) \, d\xi = \int_0^t \|\partial_t u(\xi)\|^2 \, d\xi, \quad \int_0^t (\nabla u, \nabla \partial_t u) \, d\xi = \frac{1}{2} \|\nabla u(t)\|^2 - \frac{1}{2} \|\nabla u_0\|^2.$$

For the third one we first integrate by parts,

$$\int_0^t (g, \partial_t u)_\Gamma d\xi = (g(t), u(t))_\Gamma - (g(0), u_0)_\Gamma - \int_0^t (\partial_t g, u)_\Gamma d\xi$$

and get

$$\begin{aligned} \left| \int_0^t (g, \partial_t u)_\Gamma d\xi \right| &\leq \|g(t)\|_\Gamma \|u(t)\|_\Gamma + \|g(0)\|_\Gamma \|u_0\|_\Gamma + \int_0^t \|\partial_t g\|_\Gamma \|u\|_\Gamma d\xi \\ &\leq C_\varepsilon + \varepsilon \|u\|_{H^1(\Omega)}^2 + C \int_0^t \|u\|_{H^1(\Omega)}^2 d\xi \end{aligned}$$

by Cauchy's inequality, the trace theorem and Young's inequality. The fourth term is easily bounded by

$$\left| \int_0^t K(h, \partial_t u) d\xi \right| \leq \int_0^t |K| \|h\| \|\partial_t u\| d\xi \leq C_\varepsilon \int_0^t \|h\|^2 d\xi + \varepsilon \int_0^t \|\partial_t u\|^2 d\xi,$$

as K is bounded, see Proposition 2.1. The last term in the left-hand side of (6) can be estimated using (5) and Proposition 2.2 as follows

$$\left| \int_0^t (K * u, \partial_t u) d\xi \right| \leq \int_0^t \|K * u\| \|\partial_t u\| d\xi \leq C_\varepsilon \int_0^t \|K * u\|^2 d\xi + \varepsilon \int_0^t \|\partial_t u\|^2 d\xi \leq C_\varepsilon + \varepsilon \int_0^t \|\partial_t u\|^2 d\xi.$$

For the right-hand side of (6) we deduce that

$$\left| \int_0^t (f(u, \nabla u), \partial_t u) d\xi \right| \leq \int_0^t \|f(u, \nabla u)\| \|\partial_t u\| d\xi \leq C_\varepsilon + \varepsilon \int_0^t \|\partial_t u\|^2 d\xi,$$

as f is bounded.

Putting things together we arrive at

$$\left(\frac{1}{2} - \varepsilon \right) \|\nabla u(t)\|^2 + (1 - \varepsilon) \int_0^t \|\partial_t u(\xi)\|^2 d\xi \leq C_\varepsilon + C \int_0^t \|\nabla u\|^2 d\xi,$$

which is valid for any $t \in [0, T]$. Fixing a suitable $\varepsilon > 0$ we conclude the proof by Grönwall's lemma. \square

Proposition 2.4. *Let the conditions of Proposition 2.1 be satisfied. Moreover assume that $g \in C^1([0, T], L^2(\Gamma))$, $h \in C([0, T], H^1(\Omega))$, f is Lipschitz continuous in all variables, and $u_0 \in H^2(\Omega)$. If $\langle u, K \rangle$ is a solution of (1) and (2), then there exists $C > 0$ such that*

$$\begin{aligned} (i) \quad &\max_{t \in [0, T]} \|\Delta u(t)\|^2 + \int_0^T \|\nabla \partial_t u\|^2 d\xi \leq C \\ (ii) \quad &\max_{t \in [0, T]} \|\partial_t u(t)\| \leq C. \end{aligned}$$

Proof. (i) If we set $\phi = -\Delta \partial_t u$ in (3) and integrate in time we obtain

$$- \int_0^t (\partial_t u, \Delta \partial_t u) d\xi + \int_0^t (\Delta u, \Delta \partial_t u) d\xi - \int_0^t K(h, \Delta \partial_t u) d\xi - \int_0^t (K * u, \Delta \partial_t u) d\xi = - \int_0^t (f(u, \nabla u), \Delta \partial_t u) d\xi. \quad (7)$$

The first two terms can be rewritten as

$$- \int_0^t (\partial_t u, \Delta \partial_t u) d\xi = \int_0^t \|\nabla \partial_t u\|^2 d\xi + \int_0^t (\partial_t u, \partial_t g)_\Gamma d\xi, \quad \int_0^t (\Delta u, \Delta \partial_t u) d\xi = \frac{1}{2} \|\Delta u(t)\|^2 - \frac{1}{2} \|\Delta u_0\|^2.$$

Making use of the Cauchy, Young inequalities, the trace theorem and Proposition 2.3 we deduce that

$$\begin{aligned} \left| \int_0^t (\partial_t u, \partial_t g)_\Gamma d\xi \right| &\leq \int_0^t \|\partial_t u\|_\Gamma \|\partial_t g\|_\Gamma d\xi \\ &\leq \varepsilon \int_0^t \|\partial_t u\|_\Gamma^2 d\xi + C_\varepsilon \int_0^t \|\partial_t g\|_\Gamma^2 d\xi \\ &\leq \varepsilon \int_0^t \|\partial_t u\|_{H^1(\Omega)}^2 d\xi + C_\varepsilon \\ &\leq \varepsilon \int_0^t \|\nabla \partial_t u\|^2 d\xi + C_\varepsilon. \end{aligned}$$

For the third term in (7) we first use the Green formula

$$- \int_0^t K(h, \Delta \partial_t u) d\xi = \int_0^t K[(\nabla h, \nabla \partial_t u) - (h, \partial_t g)_\Gamma] d\xi$$

and get by Cauchy's and Young's inequality

$$\left| \int_0^t K(h, \Delta \partial_t u) d\xi \right| \leq C_\varepsilon \int_0^t \|\nabla h\|^2 d\xi + \varepsilon \int_0^t \|\nabla \partial_t u\|^2 d\xi + C \int_0^t (\|h\|_\Gamma^2 + \|\partial_t g\|_\Gamma^2) d\xi \leq C_\varepsilon + \varepsilon \int_0^t \|\nabla \partial_t u\|^2 d\xi$$

as K (see Proposition 2.1) is bounded and $\|h\|_\Gamma^2$ is finite by the trace theorem. The last term in the left-hand side of (7) is rewritten as

$$- \int_0^t (K * u, \Delta \partial_t u) d\xi = \int_0^t [(K * \nabla u, \nabla \partial_t u) - (K * u, \partial_t g)_\Gamma] d\xi,$$

which gives

$$\begin{aligned} \left| \int_0^t (K * u, \Delta \partial_t u) d\xi \right| &\leq \int_0^t \|K * \nabla u\| \|\nabla \partial_t u\| d\xi + \int_0^t \|K * u\|_\Gamma \|\partial_t g\|_\Gamma d\xi \\ &\leq C_\varepsilon \int_0^t \|K * \nabla u\|^2 d\xi + \varepsilon \int_0^t \|\nabla \partial_t u\|^2 d\xi + C \int_0^t \|K * u\|_\Gamma^2 d\xi + C \int_0^t \|\partial_t g\|_\Gamma^2 d\xi \\ &\leq C_\varepsilon + \varepsilon \int_0^t \|\nabla \partial_t u\|^2 d\xi \end{aligned}$$

as $\|(K * \nabla u)(t)\|^2 \leq C \int_0^t \|\nabla u\|^2 ds$ and $\|(K * u)(t)\|_\Gamma^2 \leq C \int_0^t \|u\|_\Gamma^2 ds$, like in (5). The right-hand side of (7) is rewritten by integrating by parts as

$$- \int_0^t (f(u, \nabla u), \Delta \partial_t u) d\xi = \int_0^t (\partial_t f(u, \nabla u), \Delta u) d\xi + (f(u(0), \nabla u(0)), \Delta u(0)) - (f(u(t), \nabla u(t)), \Delta u(t))$$

so

$$\begin{aligned} \left| \int_0^t (f(u, \nabla u), \Delta \partial_t u) d\xi \right| &\leq \varepsilon \int_0^t \|\partial_t f(u, \nabla u)\|^2 d\xi + C_\varepsilon \int_0^t \|\Delta u\|^2 d\xi + C_\varepsilon + \varepsilon \|\Delta u(t)\|^2 \\ &\leq \varepsilon \int_0^t \|\partial_t \nabla u\|^2 d\xi + C_\varepsilon \int_0^t \|\Delta u\|^2 d\xi + C_\varepsilon + \varepsilon \|\Delta u(t)\|^2, \end{aligned}$$

as $u_0 \in H^2(\Omega)$, $\partial_t f(u, \nabla u) = \nabla f(u, \nabla u) \cdot \langle \partial_t u, \partial_t \nabla u \rangle$, f is Lipschitz in all variables and $\int_0^T \|\partial_t u\|^2 ds$ is bounded by Proposition 2.3.

Putting all things together we obtain

$$(1 - \varepsilon) \int_0^t \|\nabla \partial_t u(\xi)\|^2 d\xi + \left(\frac{1}{2} - \varepsilon\right) \|\Delta u(t)\|^2 \leq C_\varepsilon + C_\varepsilon \int_0^t \|\Delta u\|^2 d\xi,$$

which is valid for any $t \in [0, T]$. Fixing a sufficiently small $\varepsilon > 0$ and involving Grönwall's argument, we obtain the desired result.

(ii) The assertion follows readily from (1) and the already obtained stability results, i.e.

$$\|\partial_t u\| = \|\Delta u - Kh - K * u + f(u, \nabla u)\| \leq C \left(\max_{t \in [0, T]} |K(t)| \right) (1 + \|\Delta u\| + \|u\|) \leq C.$$

□

Proposition 2.5. *Let the conditions of Proposition 2.1 be satisfied. Moreover assume that $g \in C^1([0, T], L^2(\Gamma))$, $h \in C^1([0, T], L^2(\Omega)) \cap C([0, T], H^1(\Omega))$, $m \in C^2([0, T])$, f is Lipschitz continuous in all variables, and $u_0 \in H^2(\Omega)$. If $\langle u, K \rangle$ is a solution of (1) and (2), then there exists $C > 0$ such that*

$$\int_0^T |K'(s)|^2 ds \leq C.$$

Proof. We take the time derivative of (MP) and it follows that for any time $t \in [0, T]$ it holds

$$m'' + (\partial_t g, 1)_\Gamma + K'(h, 1) + K(\partial_t h, 1) + Km(0) + K * m' = (\nabla f(u, \nabla u) \cdot \langle \partial_t u, \partial_t \nabla u \rangle, 1). \quad (\text{MP}')$$

From this we infer

$$|(h, 1)| |K'(t)| \leq |(\nabla f(u, \nabla u) \cdot \langle \partial_t u, \partial_t \nabla u \rangle, 1)| + |K * m'| + C$$

as K is bounded, $\partial_t h \in C([0, T], L^2(\Omega))$, $\partial_t g \in C([0, T], L^2(\Gamma))$ and $m \in C^2([0, T])$. Since f is Lipschitz continuous in all variables and $\partial_t u$ is $L^2(\Omega)$ -bounded we obtain

$$\omega |K'(t)| \leq |(h, 1)| |K'| \leq C + C \|\partial_t \nabla u\|.$$

Taking square and integrating in time we arrive at

$$\int_0^T |K'(\xi)|^2 d\xi \leq C + C \int_0^T \|\partial_t \nabla u(\xi)\|^2 d\xi \leq C.$$

□

Uniqueness. Now, we are in a position to state unicity of solution. Suppose $\langle u_1, K_1 \rangle$ and $\langle u_2, K_2 \rangle$ solve (P)-(MP), then by subtracting the corresponding variational formulations from each other we obtain

$$(\partial_t(u_1 - u_2), \phi) + (\nabla(u_1 - u_2), \nabla \phi) + (K_1(t) - K_2(t))(h, \phi) + (K_1 * u_1 - K_2 * u_2, \phi) = (f(u_1, \nabla u_1) - f(u_2, \nabla u_2), \phi),$$

$$(K_1(t) - K_2(t))(h, 1) + (K_1 - K_2) * m = (f(u_1, \nabla u_1) - f(u_2, \nabla u_2), 1).$$

This we rewrite using $e_K(t) = K_1(t) - K_2(t)$ and $e_u(\mathbf{x}, t) = u_1(\mathbf{x}, t) - u_2(\mathbf{x}, t)$

$$\begin{cases} (\partial_t e_u, \phi) + (\nabla e_u, \nabla \phi) + e_K(h, \phi) + (K_1 * e_u, \phi) + (e_K * u_2, \phi) = (f(u_1, \nabla u_1) - f(u_2, \nabla u_2), \phi) \\ e_K(h, 1) + e_K * m = (f(u_1, \nabla u_1) - f(u_2, \nabla u_2), 1). \end{cases} \quad \begin{matrix} (8a) \\ (8b) \end{matrix}$$

Theorem 1. *Assume that $h \in C([0, T], L^2(\Omega))$, $\min_{t \in [0, T]} |(h(t), 1)| \geq \omega > 0$ and $m \in C([0, T])$. The function f is supposed to be Lipschitz continuous in all variables. Then the problem (P)-(MP) has at most one solution $\langle u, K \rangle \in L^2((0, T), H^1(\Omega)) \times L^2(0, T)$ with $\partial_t u \in L^2((0, T), (H^1(\Omega))^*)$.*

Proof. The Lipschitz continuity of f , Grönwall's lemma and (8b) implies

$$|e_K(t)| \leq C \|e_u(t)\|_{H^1(\Omega)} + C \int_0^t \|e_u\|_{H^1(\Omega)} d\xi. \quad (9)$$

We put $\phi = e_u$ in (8a) and integrate in time

$$\begin{aligned} & \frac{1}{2} \|e_u(t)\|^2 + \int_0^t \|\nabla e_u\|^2 d\xi + \int_0^t e_K(h, e_u) d\xi + \int_0^t (K_1 * e_u, e_u) d\xi + \int_0^t (e_K * u_2, e_u) d\xi \\ &= \int_0^t (f(u_1, \nabla u_1) - f(u_2, \nabla u_2), e_u) d\xi. \end{aligned}$$

Using Cauchy's inequality, we obtain successively the bounds

$$\int_0^t \|f(u_1, \nabla u_1) - f(u_2, \nabla u_2)\| \|e_u\| d\xi \leq \varepsilon \int_0^t \|\nabla e_u\|^2 d\xi + C_\varepsilon \int_0^t \|e_u\|^2 d\xi,$$

as f is Lipschitz,

$$\int_0^t \|e_K * u_2\| \|e_u\| d\xi \leq \varepsilon \int_0^t e_K^2 d\xi + C_\varepsilon \int_0^t \|e_u\|^2 d\xi$$

as $u_2 \in C([0, T], L^2(\Omega))$, which follows from $\partial_t u_2 \in L^2((0, T), L^2(\Omega))$,

$$\int_0^t \|K_1 * e_u\| \|e_u\| d\xi \leq C \int_0^t \|e_u\|^2 d\xi,$$

as $K_1 \in L^2(0, T)$, and using $h \in C([0, T], L^2(\Omega))$

$$\int_0^t |e_K| |h| \|e_u\| d\xi \leq \varepsilon \int_0^t |e_K|^2 d\xi + C_\varepsilon \int_0^t \|e_u\|^2 d\xi \leq \varepsilon \int_0^t \|e_u\|_{H^1(\Omega)}^2 d\xi + C_\varepsilon \int_0^t \|e_u\|^2 d\xi.$$

From these estimates we obtain

$$\|e_u(t)\|^2 + (1 - \varepsilon) \int_0^t \|\nabla e_u\|^2 d\xi \leq C_\varepsilon \int_0^t \|e_u\|^2 d\xi,$$

and conclude that $\max_{t \in [0, T]} \|e_u(t)\|^2 + \int_0^T \|\nabla e_u\|^2 d\xi = 0$ by Grönwall's lemma when fixing a suitable $\varepsilon > 0$. So u is unique in $C([0, T], L^2(\Omega)) \cap L^2((0, T), H^1(\Omega))$. The uniqueness of K in $L^2(0, T)$ follows from (9). \square

3. Time discretization, existence of a solution

Rothe's method [19, 18] represents a constructive method suitable for solving evolution problems. Using a simple discretization in time, a time-dependent problem is approximated by a sequence of elliptic problems which have to be solved successively with increasing time step. This standard technique is in our case complicated by the unknown convolution kernel K . There exists a simple way to overcome this difficulty.

For ease of explanation we consider an equidistant time-partitioning of the time frame $[0, T]$ with a step $\tau = T/n$, for any $n \in \mathbb{N}$. We use the notation $t_i = i\tau$ and for any function z we write

$$z_i = z(t_i), \quad \delta z_i = \frac{z_i - z_{i-1}}{\tau}.$$

We will consider a decoupled system with unknowns $\langle u_i, K_i \rangle$ for $i = 1, \dots, n$. At time t_i we infer from (3) the backward Euler scheme

$$(\delta u_i, \phi) - (\Delta u_i, \phi) + K_i(h_i, \phi) + \left(\sum_{k=1}^i K_k u_{i-k} \tau, \phi \right) = (f_{i-1}, \phi). \quad (10)$$

where $f_i = f(u_i, \nabla u_i)$. Like (P) and (MP) one obtains for $\phi \in H^1(\Omega)$ that

$$(\delta u_i, \phi) + (\nabla u_i, \nabla \phi) + (g_i, \phi)_\Gamma + K_i(h_i, \phi) + \left(\sum_{k=1}^i K_k u_{i-k} \tau, \phi \right) = (f_{i-1}, \phi) \quad (\text{DPi})$$

and

$$m'_i + (g_i, 1)_\Gamma + K_i(h_i, 1) + \sum_{k=1}^i K_k m_{i-k} \tau = (f_{i-1}, 1). \quad (\text{DMPi})$$

Note that for a given $i \in \{1, \dots, n\}$ we solve first (DMPi) and then (DPi). Further we increase i to $i+1$.

Proposition 3.1. *Let f be bounded, i.e. $|f| \leq C$. Moreover assume that $g \in C([0, T], L^2(\Gamma))$, $h \in C([0, T], L^2(\Omega))$, $\min_{t \in [0, T]} |h(t, 1)| \geq \omega > 0$, $u_0 \in H^1(\Omega)$ and $m \in C^1([0, T])$. Then there exist $C > 0$ and $\tau_0 > 0$ such that for any $\tau < \tau_0$ and each $i \in \{1, \dots, n\}$ we have*

(i) *there exist $K_i \in \mathbb{R}$ and $u_i \in H^1(\Omega)$ obeying (DMPi) and (DPi)*

(ii) $\max_{1 \leq i \leq n} |K_i| \leq C$.

Proof. (i) Set $\tau_0 = \min \left\{ 1, \frac{\omega}{2|m_0|} \right\}$. Then for any $\tau < \tau_0$ we may write by triangle inequality that

$$0 < \omega - |m_0| \tau_0 \leq \omega - |m_0| \tau \leq |(h_i, 1)| - |m_0| \tau \leq |(h_i, 1) - m_0 \tau|$$

We apply the following recursive deduction for $i = 1, \dots, n$.

Step 1: Let $u_{i-1} \in H^1(\Omega)$ be given. Then (DMPi) implies the existence of $K_i \in \mathbb{R}$ such that

$$K_i [(h_i, 1) - m_0 \tau] = (f_{i-1}, 1) - m'_i - (g_i, 1)_\Gamma - \sum_{k=1}^{i-1} K_k m_{i-k} \tau. \quad (11)$$

Step 2: The existence of $u_i \in H^1(\Omega)$ follows from (DPi) by the Lax-Milgram lemma.

(ii) The relation (11) yields

$$|K_i| \leq C \left(1 + \sum_{k=1}^{i-1} |K_k| \tau \right),$$

which is valid for any $i = 1, \dots, n$. An application of the discrete Grönwall lemma gives the uniform bound of $|K_i|$. \square

Proposition 3.2. *Let the conditions of Proposition 3.1 be satisfied. Then there exists $C > 0$ such that for any $\tau < \tau_0$*

$$\max_{1 \leq j \leq n} \|u_j\|^2 + \sum_{i=1}^n \|\nabla u_i\|^2 \tau + \sum_{i=1}^n \|u_i - u_{i-1}\|^2 \leq C.$$

Proof. If we set $\phi = u_i \tau$ in (DPi) and sum up for $i = 1, \dots, j$ we obtain

$$\sum_{i=1}^j (\delta u_i, u_i) \tau + \sum_{i=1}^j \|\nabla u_i\|^2 \tau + \sum_{i=1}^j (g_i, u_i)_\Gamma \tau + \sum_{i=1}^j K_i (h_i, u_i) \tau + \sum_{i=1}^j \sum_{k=1}^i (K_k u_{i-k} \tau, u_i) \tau = \sum_{i=1}^j (f_{i-1}, u_i) \tau. \quad (12)$$

The summation by parts formula says that

$$\sum_{i=1}^j (\delta u_i, u_i) \tau = \sum_{i=1}^j (u_i - u_{i-1}, u_i) = \frac{1}{2} \left(\|u_j\|^2 - \|u_0\|^2 + \sum_{i=1}^j \|u_i - u_{i-1}\|^2 \right).$$

For the third term of (12) we get

$$\left| \sum_{i=1}^j (g_i, u_i)_{\Gamma} \tau \right| \leq \sum_{i=1}^j \|g_i\|_{\Gamma} \|u_i\|_{\Gamma} \tau \leq C \sum_{i=1}^j \|g_i\|_{\Gamma} \|u_i\|_{H^1(\Omega)} \tau \leq C_{\varepsilon} \sum_{i=1}^j \|g_i\|_{\Gamma}^2 \tau + \varepsilon \sum_{i=1}^j \|u_i\|_{H^1(\Omega)}^2 \tau$$

by Cauchy's inequality, the trace theorem and Young's inequality. The fourth term in (12) is easily bounded by

$$\left| \sum_{i=1}^j K_i(h_i, u_i) \tau \right| \leq \sum_{i=1}^j |K_i| \|h_i\| \|u_i\| \tau \leq C \sum_{i=1}^j \|h_i\|^2 \tau + C \sum_{i=1}^j \|u_i\|^2 \tau,$$

as K_i is bounded, see Proposition 3.1. The last term in the left-hand side of (12) is

$$\left| \sum_{i=1}^j \sum_{k=1}^i (K_k u_{i-k}, u_i) \tau^2 \right| \leq \sum_{i=1}^j \sum_{k=1}^i |(K_k u_{i-k}, u_i)| \tau^2 \leq C \sum_{i=1}^j \sum_{k=1}^i \|u_{i-k}\|^2 \tau^2 + C \sum_{i=1}^j \sum_{k=1}^i \|u_i\|^2 \tau^2 \leq C \sum_{i=0}^j \|u_i\|^2 \tau,$$

again as K_i is bounded, see Proposition 3.1. The right-hand side of (12) can be estimated as follows

$$\left| \sum_{i=1}^j (f_{i-1}, u_i) \tau \right| \leq \sum_{i=1}^j \|f_{i-1}\| \|u_i\| \tau \leq C + C \sum_{i=1}^j \|u_i\|^2 \tau,$$

as f is bounded.

Putting all things together we obtain

$$\|u_j\|^2 + \sum_{i=1}^k \|u_i - u_{i-1}\|^2 + (1 - \varepsilon) \sum_{i=1}^j \|\nabla u_i\|^2 \tau \leq C_{\varepsilon} + C \sum_{i=1}^j \|u_i\|^2 \tau.$$

Fixing a sufficiently small $\varepsilon > 0$ and involving the discrete Grönwall lemma we conclude the proof. \square

Proposition 3.3. *Let the conditions of Proposition 3.1 be satisfied. Moreover suppose that $g \in C^1([0, T], L^2(\Gamma))$. Then there exists $C > 0$ such that for any $\tau < \tau_0$ it holds*

$$\max_{1 \leq j \leq n} \|\nabla u_j\|^2 + \sum_{i=1}^n \|\delta u_i\|^2 \tau + \sum_{i=1}^n \|\nabla u_i - \nabla u_{i-1}\|^2 \leq C.$$

Proof. If we set $\phi = \delta u_i \tau$ in (DPi) and sum up for $i = 1, \dots, j$ we obtain

$$\sum_{i=1}^j \|\delta u_i\|^2 \tau + \sum_{i=1}^j (\nabla u_i, \nabla \delta u_i) \tau + \sum_{i=1}^j (g_i, \delta u_i)_{\Gamma} \tau + \sum_{i=1}^j K_i(h_i, \delta u_i) \tau + \sum_{i=1}^j \sum_{k=1}^i (K_k u_{i-k}, \delta u_i) \tau^2 = \sum_{i=1}^j (f_{i-1}, \delta u_i) \tau. \quad (13)$$

The second term can be rewritten as

$$\sum_{i=1}^j (\nabla u_i, \nabla \delta u_i) \tau = \sum_{i=1}^j (\nabla u_i, \nabla u_i - \nabla u_{i-1}) = \frac{1}{2} \left(\|\nabla u_j\|^2 - \|\nabla u_0\|^2 + \sum_{i=1}^k \|\nabla u_i - \nabla u_{i-1}\|^2 \right).$$

For the third one we first use summation by parts,

$$\sum_{i=1}^j (g_i, u_i - u_{i-1})_{\Gamma} = (g_j, u_j)_{\Gamma} - (g_0, u_0)_{\Gamma} - \sum_{i=1}^j (g_i - g_{i-1}, u_i)_{\Gamma}$$

and get

$$\begin{aligned} \left| \sum_{i=1}^j (g_i, \delta u_i)_{\Gamma} \tau \right| &\leq \|g_j\|_{\Gamma} \|u_j\|_{\Gamma} + \|g_0\|_{\Gamma} \|u_0\|_{\Gamma} + \sum_{i=1}^j \|\delta g_i\|_{\Gamma} \|u_i\|_{\Gamma} \tau \\ &\leq C_{\varepsilon} + \varepsilon \|u_j\|_{H^1(\Omega)}^2 + C \sum_{i=1}^j \|u_i\|_{H^1(\Omega)}^2 \tau \\ &\leq C_{\varepsilon} + \varepsilon \|\nabla u_j\|^2 \end{aligned}$$

by Cauchy's inequality, the trace theorem, Young's inequality, and Proposition 3.2. The fourth term in (13) is easily bounded by

$$\left| \sum_{i=1}^j K_i(h_i, \delta u_i) \tau \right| \leq \sum_{i=1}^j |K_i| \|h_i\| \|\delta u_i\| \tau \leq C_\varepsilon \sum_{i=1}^j \|h_i\|^2 \tau + \varepsilon \sum_{i=1}^j \|\delta u_i\|^2 \tau \leq C_\varepsilon + \varepsilon \sum_{i=1}^j \|\delta u_i\|^2 \tau,$$

as K is bounded, see Proposition 3.1. The last term in the left-hand side of (13) can be estimated as follows

$$\begin{aligned} \left| \sum_{i=1}^j \sum_{k=1}^i (K_k u_{i-k}, \delta u_i) \tau^2 \right| &\leq \sum_{i=1}^j \sum_{k=1}^i |K_k| \|u_{i-k}\| \|\delta u_i\| \tau^2 \\ &\leq \sum_{i=1}^j \sum_{k=1}^i (C_\varepsilon \|u_{i-k}\|^2 + \varepsilon \|\delta u_i\|^2) \tau^2 \\ &\leq C_\varepsilon + \varepsilon \sum_{i=1}^j \|\delta u_i\|^2 \tau \end{aligned}$$

using Propositions 3.1 and 3.2. The right-hand side of (13) can be enlarged by

$$\left| \sum_{i=1}^j (f_{i-1}, \delta u_i) \tau \right| \leq \sum_{i=1}^j \|f_{i-1}\| \|\delta u_i\| \tau \leq C_\varepsilon + \varepsilon \sum_{i=1}^j \|\delta u_i\|^2 \tau$$

as f is bounded.

Putting all things together we obtain

$$(1 - \varepsilon) \sum_{i=1}^j \|\delta u_i\|^2 \tau + \left(\frac{1}{2} - \varepsilon\right) \|\nabla u_j\|^2 + \frac{1}{2} \sum_{i=1}^k \|\nabla u_i - \nabla u_{i-1}\|^2 \leq C_\varepsilon.$$

Fixing a suitable $\varepsilon > 0$ we conclude the proof. \square

Inspecting the relation (DPi) we may write for any $\phi \in H^1(\Omega)$ that

$$(-\Delta u_i, \phi) = (\nabla u_i, \nabla \phi) + (g_i, \phi)_\Gamma = (f_{i-1}, \phi) - (\delta u_i, \phi) - K_i(h_i, \phi) - \left(\sum_{k=1}^i K_k u_{i-k} \tau, \phi \right). \quad (14)$$

The term $-\Delta u_i$ has to be understood in the sense of duality, as a functional on $H^1(\Omega)$. The right-hand side of (14) can be estimated by $C(1 + \|\delta u_i\|) \|\phi\|$. Thus there exists an extension of $-\Delta u_i$ to $L^2(\Omega)$ according to Hahn-Banach theorem, cf. [22, p. 173]. This extension will have the same norm as the functional on $H^1(\Omega)$. Therefore taking into account the assumptions and the stability results from Proposition 3.3 we immediately obtain

$$\sum_{i=1}^n \|\Delta u_i\|^2 \tau \leq C + C \sum_{i=1}^n \|\delta u_i\|^2 \tau \leq C. \quad (15)$$

Proposition 3.4. Assume that $g \in C^1([0, T], L^2(\Gamma))$, $h \in C([0, T], H^1(\Omega))$, $\min_{t \in [0, T]} |(h(t), 1)| \geq \omega > 0$, $u_0 \in H^2(\Omega)$ and $m \in C^1([0, T])$. The function f is supposed to be bounded, i.e. $|f| \leq C$, and Lipschitz continuous in all variables. Then there exist $C > 0$ such that for any $\tau < \tau_0$ we have

- (i) $\max_{1 \leq j \leq n} \|\Delta u_j\|^2 + \sum_{i=1}^n \|\nabla \delta u_i\|^2 \tau + \sum_{i=1}^n \|\Delta u_i - \Delta u_{i-1}\|^2 \leq C$
- (ii) $\max_{1 \leq j \leq n} \|\delta u_j\| \leq C.$

Proof. (i) If we set $\phi = -\Delta\delta u_i\tau$ in (10) and sum up for $i = 1, \dots, j$ we obtain

$$-\sum_{i=1}^j (\delta u_i, \Delta\delta u_i)\tau + \sum_{i=1}^j (\Delta u_i, \Delta\delta u_i)\tau - \sum_{i=1}^j K_i(h_i, \Delta\delta u_i)\tau - \sum_{i=1}^j \sum_{k=1}^i (K_k u_{i-k}, \Delta\delta u_i)\tau^2 = -\sum_{i=1}^j (f_{i-1}, \Delta\delta u_i)\tau. \quad (16)$$

The first two terms can be rewritten as

$$-\sum_{i=1}^j (\delta u_i, \Delta\delta u_i)\tau = \sum_{i=1}^j \|\nabla\delta u_i\|^2 \tau + \sum_{i=1}^j (\delta u_i, \delta g_i)\tau, \quad \sum_{i=1}^j (\Delta u_i, \Delta\delta u_i)\tau = \frac{1}{2} \left(\|\Delta u_j\|^2 - \|\Delta u_0\|^2 + \sum_{i=1}^j \|\Delta u_i - \Delta u_{i-1}\|^2 \right).$$

Using the Cauchy and Young inequalities, the trace theorem and Proposition 3.3 we successively deduce that

$$\begin{aligned} \left| \sum_{i=1}^j (\delta u_i, \delta g_i)\tau \right| &\leq \sum_{i=1}^j \|\delta u_i\|_{\Gamma} \|\delta g_i\|_{\Gamma} \tau \\ &\leq \varepsilon \sum_{i=1}^j \|\delta u_i\|_{\Gamma}^2 \tau + C_{\varepsilon} \sum_{i=1}^j \|\delta g_i\|_{\Gamma}^2 \tau \\ &\leq \varepsilon \sum_{i=1}^j \|\delta u_i\|_{H^1(\Omega)}^2 \tau + C_{\varepsilon} \\ &\leq \varepsilon \sum_{i=1}^j \|\delta \nabla u_i\|^2 \tau + C_{\varepsilon}. \end{aligned}$$

For the third term in (16) we first integrate by parts,

$$-K_i(h_i, \Delta\delta u_i)\tau = K_i[(\nabla h_i, \nabla\delta u_i) - (h_i, \delta g_i)_{\Gamma}] \tau$$

and get by Cauchy's and Young's inequality

$$\left| \sum_{i=1}^j K_i(h_i, \Delta\delta u_i)\tau \right| \leq C_{\varepsilon} \sum_{i=1}^j \|\nabla h_i\|^2 \tau + \varepsilon \sum_{i=1}^j \|\nabla\delta u_i\|^2 \tau + C \sum_{i=1}^j (\|h_i\|_{\Gamma}^2 + \|\delta g_i\|_{\Gamma}^2) \tau \leq C_{\varepsilon} + \varepsilon \sum_{i=1}^j \|\nabla\delta u_i\|^2 \tau$$

as K_i (see Proposition 3.1) is bounded, $h_i \in H^1(\Omega)$ and $\|h_i\|_{\Gamma}^2$ is finite by the trace theorem. The last term in the left-hand side of (16) is rewritten as

$$-\sum_{i=1}^j \sum_{k=1}^i (K_k u_{i-k}, \Delta\delta u_i)\tau^2 = \sum_{i=1}^j \sum_{k=1}^i [(K_k \nabla u_{i-k}, \nabla\delta u_i) - (K_k u_{i-k}, \delta g_i)_{\Gamma}] \tau^2,$$

which gives

$$\begin{aligned} \left| \sum_{i=1}^j \sum_{k=1}^i (K_k u_{i-k}, \Delta\delta u_i)\tau^2 \right| &\leq \sum_{i=1}^j \sum_{k=1}^i |K_k| \|\nabla u_{i-k}\| \|\nabla\delta u_i\| \tau^2 + \sum_{i=1}^j \sum_{k=1}^i |K_k| \|u_{i-k}\|_{\Gamma} \|\delta g_i\|_{\Gamma} \tau^2 \\ &\leq C \sum_{i=1}^j \|\nabla\delta u_i\| \tau + C \sum_{i=1}^j \|\delta g_i\|_{\Gamma} \tau \\ &\leq C_{\varepsilon} + \varepsilon \sum_{i=1}^j \|\nabla\delta u_i\|^2 \tau \end{aligned}$$

as Proposition 3.1, the trace theorem, u_i and ∇u_i are $L^2(\Omega)$ -bounded (Proposition 3.3). The right-hand side of (16) is rewritten by summation by parts as

$$\sum_{i=1}^j (f_{i-1}, \delta\Delta u_i)\tau = \sum_{i=1}^j (f_{i-1}, \Delta u_i - \Delta u_{i-1}) = (f_{j-1}, \Delta u_j) - (f_0, \Delta u_0) - \sum_{i=1}^{j-1} (\delta f_i, \Delta u_i)\tau$$

so

$$\begin{aligned}
 \left| \sum_{i=1}^j (f_{i-1}, \Delta \delta u_i \tau) \right| &\leq C_\varepsilon + \varepsilon \|\Delta u_j\|^2 + \varepsilon \sum_{i=1}^{j-1} \|\delta f_i\|^2 \tau + C_\varepsilon \sum_{i=1}^{j-1} \|\Delta u_i\|^2 \tau \\
 &\leq C_\varepsilon + \varepsilon \|\Delta u_j\|^2 + \varepsilon \sum_{i=1}^{j-1} \|\delta u_i\|^2 \tau + \varepsilon \sum_{i=1}^{j-1} \|\delta \nabla u_i\|^2 \tau + C_\varepsilon \sum_{i=1}^{j-1} \|\Delta u_i\|^2 \tau \\
 &\leq C_\varepsilon + \varepsilon \|\Delta u_j\|^2 + \varepsilon \sum_{i=1}^{j-1} \|\delta \nabla u_i\|^2 \tau + C_\varepsilon \sum_{i=1}^{j-1} \|\Delta u_i\|^2 \tau
 \end{aligned}$$

as $u_0 \in H^2(\Omega)$, $\delta f_i = \nabla f_i \cdot \langle \delta u_i, \delta \nabla u_i \rangle$ and f is Lipschitz.

Putting all things together we arrive at

$$(1 - \varepsilon) \sum_{i=1}^j \|\delta \nabla u_i\|^2 \tau + \left(\frac{1}{2} - \varepsilon\right) \|\Delta u_j\|^2 + \frac{1}{2} \sum_{i=1}^k \|\Delta u_i - \Delta u_{i-1}\|^2 \leq C_\varepsilon + C_\varepsilon \sum_{i=1}^j \|\Delta u_i\|^2 \tau.$$

Choosing a suitable $\varepsilon > 0$ we close the proof by Grönwall's argument.

(ii) The relation (14) gives for any $\phi \in H^1(\Omega)$

$$(\delta u_i, \phi) = (f_{i-1}, \phi) - K_i(h_i, \phi) - \left(\sum_{k=1}^i K_k u_{i-k} \tau, \phi \right) + (\Delta u_i, \phi).$$

The stability results from Propositions 3.1–3.4(i) ensure that the right-hand side can be seen as a linear bounded functional on $L^2(\Omega)$. Thus, the left-hand side allows extension from $H^1(\Omega)$ to $L^2(\Omega)$ with the same norm estimate through the Hahn-Banach theorem (cf. the deduction after (14)), i.e.,

$$\|\delta u_i\| = \sup_{\|\phi\| \leq 1} |(\delta u_i, \phi)| \leq \left\| \Delta u_i - K_i h_i - \sum_{k=1}^i K_k u_{i-k} \tau + f_{i-1} \right\| \leq C.$$

□

Proposition 3.5. Assume that $g \in C^1([0, T], L^2(\Gamma))$, $h \in C([0, T], H^1(\Omega)) \cap C^1([0, T], L^2(\Omega))$, $\min_{t \in [0, T]} |(h(t), 1)| \geq \omega > 0$, $u_0 \in H^2(\Omega)$ and $m \in C^2([0, T])$. The function f is supposed to be bounded, i.e. $|f| \leq C$, and Lipschitz continuous in all variables. Then there exist $C > 0$ such that for any $\tau < \tau_0$ we have

$$\sum_{i=1}^j |\delta K_i|^2 \tau \leq C.$$

Proof. The fact that $u_0 \in H^2(\Omega)$ implies that the PDE from (1) is fulfilled at $t = 0$, i.e. one can define the initial value for $\partial_t u$ in the following way

$$\partial_t u(0) := f(u_0, \nabla u_0) + \Delta u_0 - K(0)h(0) \in L^2(\Omega).$$

Applying measurement to this equation gives

$$m'_0 + (g_0, 1)_\Gamma + K_0(h_0, 1) = (f_0, 1). \quad (\text{DMP0})$$

We would like to apply the δ -operator to (DMP*i*). Using the rule $\delta(a_i b_i) = \delta a_i b_i + a_{i-1} \delta b_i$ we get for $i \geq 2$

$$\delta m'_i + (\delta g_i, 1)_\Gamma + \delta K_i(h_i, 1) + K_{i-1}(\delta h_i, 1) + K_i m_0 + \sum_{k=1}^{i-1} K_k \delta m_{i-k} \tau = (\delta f_{i-1}, 1).$$

Thus for $i \geq 2$ it holds

$$|\delta K_i| |(h_i, 1)| \leq |\delta m'_i| + |(\delta g_i, 1)_\Gamma| + |K_{i-1}(\delta h_i, 1)| + |K_i m_0| + \sum_{k=1}^{i-1} |K_k \delta m_{i-k}| \tau + |(\delta f_{i-1}, 1)| \leq C + C (\|\delta u_{i-1}\| + \|\delta \nabla u_{i-1}\|).$$

Further, we subtract (DMP0) from (DMPi) for $i = 1$ to get

$$\delta m'_1 + (\delta g_1, 1)_\Gamma + \delta K_1 (h_1, 1) + K_0(\delta h_1, 1) + K_1 m_0 = 0 \quad (17)$$

and we estimate

$$|\delta K_1| |(h_1, 1)| \leq |K_1 m_0| + |K_0(\delta h_1, 1)| + |(\delta g_1, 1)_\Gamma| + |\delta m'_1|.$$

The proof is completed by applying Propositions 3.3 and 3.4 to

$$\sum_{i=1}^j |\delta K_i|^2 \tau \leq C + C \sum_{i=2}^j (\|\delta u_{i-1}\|^2 + \|\delta \nabla u_{i-1}\|^2) \tau \leq C$$

as $|(h_i, 1)| \geq \omega > 0$. □

4. Existence of a solution

Now, let us introduce the following piecewise linear function in time

$$u_n : [0, T] \rightarrow L^2(\Omega) : t \mapsto \begin{cases} u_0 & t = 0 \\ u_{i-1} + (t - t_{i-1})\delta u_i & t \in (t_{i-1}, t_i] \end{cases}, \quad 0 \leq i \leq n,$$

and a step function

$$\bar{u}_n : [0, T] \rightarrow L^2(\Omega) : t \mapsto \begin{cases} u_0 & t = 0 \\ u_i & t \in (t_{i-1}, t_i] \end{cases}, \quad 0 \leq i \leq n.$$

Similarly we define $\bar{K}_n, \bar{h}_n, \bar{g}_n, \bar{m}_n$ and \bar{m}'_n . These prolongations are also called Rothe's (piecewise linear and continuous, or piecewise constant) functions. Now, we can rewrite (DPi) and (DMPi) on the whole time frame as¹

$$(\partial_t u_n, \phi) + (\nabla \bar{u}_n, \nabla \phi) + (\bar{g}_n, \phi)_\Gamma + \bar{K}_n(\bar{h}_n, \phi) + \sum_{k=1}^{\lfloor t \rfloor_\tau} (\bar{K}_n(t_k) \bar{u}_n(t - t_k) \tau, \phi) = (f(\bar{u}_n(t - \tau), \nabla \bar{u}_n(t - \tau)), \phi). \quad (DP)$$

and

$$\bar{m}'_n + (\bar{g}_n, 1)_\Gamma + \bar{K}_n(\bar{h}_n, 1) + \sum_{k=1}^{\lfloor t \rfloor_\tau} \bar{K}_n(t_k) \bar{m}_n(t - t_k) \tau = (f(\bar{u}_n(t - \tau), \nabla \bar{u}_n(t - \tau)), 1). \quad (DMP)$$

Now, we are in a position to prove the existence of a weak solution to (P) and (MP).

Theorem 2. *Suppose the conditions of Proposition 3.5 are fulfilled. Then there exists a weak solution $\langle u, K \rangle$ to (P) and (MP), where $u \in C([0, T], H^1(\Omega))$, $\partial_t u \in L^\infty((0, T), L^2(\Omega))$, $K \in C([0, T])$, $K' \in L^2(0, T)$.*

¹ $\lfloor t \rfloor_\tau = i$ when $t \in (t_{i-1}, t_i]$

Proof. From Propositions 3.2 and 3.3 we have $\|u_j\| \leq C$ and $\sum_{i=1}^n \|\delta u_i\|^2 \tau \leq C$, which means that for all $n > 0$ it holds

$$\|\bar{u}_n(t)\|_{H^1(\Omega)} \leq C \quad \text{for all } t \in [0, T], \quad \int_0^T \|\partial_t u_n(\xi)\|^2 d\xi \leq C.$$

Using [19, Lemma 1.3.13] there exists $u \in C([0, T], L^2(\Omega)) \cap L^\infty((0, T), H^1(\Omega))$ which is time-differentiable a.e. in $[0, T]$ and a subsequence $(u_{n_k})_{k \in \mathbb{N}}$ of $(u_n)_{n \in \mathbb{N}}$ such that

$$\begin{cases} u_{n_k} \rightarrow u, & \text{in } C([0, T], L^2(\Omega)) \end{cases} \quad (18a)$$

$$\begin{cases} u_{n_k}(t) \rightarrow u(t), & \text{in } H^1(\Omega), \quad \forall t \in [0, T] \end{cases} \quad (18b)$$

$$\begin{cases} \bar{u}_{n_k}(t) \rightarrow u(t), & \text{in } H^1(\Omega), \quad \forall t \in [0, T] \end{cases} \quad (18c)$$

$$\begin{cases} \partial_t u_{n_k} \rightarrow \partial_t u, & \text{in } L^2((0, T), L^2(\Omega)) \end{cases} \quad (18d)$$

which we denote again by u_n for ease of reading. Moreover since $\|\delta u_j\| \leq C$ we have that $\partial_t u \in L^\infty((0, T), L^2(\Omega))$ and $u : [0, T] \rightarrow L^2(\Omega)$ is Lipschitz continuous, i.e. $\|u(t) - u(t')\| \leq C|t - t'|$ for all $t, t' \in [0, T]$.

Using Nečas' inequality [23], the fact that $\sum_{i=1}^n \|\nabla u_i\|^2 \tau$ is bounded (Proposition 3.2) and $u \in L^\infty((0, T), H^1(\Omega))$ we obtain

$$\int_0^T \|\bar{u}_n - u\|_\Gamma^2 d\xi \leq \varepsilon \int_0^T \|\nabla(\bar{u}_n - u)\|^2 d\xi + C_\varepsilon \int_0^T \|\bar{u}_n - u\|^2 d\xi \leq \varepsilon + C_\varepsilon \int_0^T \|\bar{u}_n - u\|^2 d\xi.$$

Passing to the limit and applying (18a) it holds

$$\lim_{n \rightarrow +\infty} \int_0^T \|\bar{u}_n - u\|_\Gamma^2 d\xi \leq \varepsilon \implies \bar{u}_n \rightarrow u, \quad \text{a.e. in } (0, T) \times \Gamma. \quad (19)$$

The trace theorem and Proposition 3.4 give

$$\begin{aligned} \|u_n(t + \varepsilon) - u_n(t)\|_\Gamma &= \left\| \int_t^{t+\varepsilon} \partial_t u_n(s) ds \right\|_\Gamma \leq \int_t^{t+\varepsilon} \|\partial_t u_n(s)\|_\Gamma ds \leq \sqrt{\varepsilon} \sqrt{\int_t^{t+\varepsilon} \|\partial_t u_n(s)\|_\Gamma^2 ds} \\ &\leq \sqrt{\varepsilon} \sqrt{\int_0^T \|\partial_t u_n(s)\|_\Gamma^2 ds} \leq C \sqrt{\varepsilon} \sqrt{\int_0^T \|\partial_t u_n(s)\|_{H^1(\Omega)}^2 ds} \leq C \sqrt{\varepsilon}. \end{aligned}$$

This together with (19) yields that $u_n \rightarrow u$ in $C([0, T], L^2(\Gamma))$. Integration by parts implies

$$\|\nabla \bar{u}_n - \nabla \bar{u}_m\|^2 = (-\Delta(\bar{u}_n - \bar{u}_m), \bar{u}_n - \bar{u}_m) - (\bar{g}_n - \bar{g}_m, \bar{u}_n - \bar{u}_m)_\Gamma.$$

The assumption $g \in C^1([0, T], L^2(\Gamma))$ ensures that $\bar{g}_n \rightarrow g$ in $C([0, T], L^2(\Gamma))$. Taking into account $\|\Delta \bar{u}_n(t)\| \leq C$, cf. Proposition 3.4, we see that \bar{u}_n is a Cauchy sequence in $C([0, T], H^1(\Omega))$. Combining this with

$$\|u_n(t) - \bar{u}_n(t)\|_{H^1(\Omega)} \leq C \sqrt{\tau} \sqrt{\int_t^{t+\tau} \|\partial_t u_n(s)\|_{H^1(\Omega)}^2 ds} \leq C \sqrt{\tau}$$

and (18a) we conclude that

$$\bar{u}_n \rightarrow u, \quad u_n \rightarrow u \quad \text{in } C([0, T], H^1(\Omega)). \quad (20)$$

Using Propositions 3.1 and 3.5 we have

$$|\bar{K}_n(t)| \leq C \quad \text{for all } t \in [0, T], \quad \int_0^T |\partial_t K_n(\xi)|^2 d\xi \leq C,$$

which means by the Arzelà-Ascoli theorem [24, Theorem 11.28] that there exists a subsequence $(K_{n_k})_{k \in \mathbb{N}}$ (which we denote by the same symbol again) that converges uniformly on $[0, T]$, say to K . The reflexivity of the space $L^2(0, T)$ implies that $\partial_t K_{n_k} \rightharpoonup \partial_t K$ in $L^2(0, T)$. It becomes trivial to see that

$$\lim_{n \rightarrow +\infty} \int_0^t \bar{K}_n(\bar{h}_n, \phi) d\xi = \int_0^t K(h, \phi) d\xi. \quad (21)$$

Applying (18a) combined with the uniform convergence of K_n we have

$$\lim_{n \rightarrow +\infty} \int_0^t \sum_{k=1}^{\lfloor t/\tau \rfloor} (\bar{K}_n(t_k) \bar{u}_n(\xi - t_k) \tau, \phi) d\xi = \int_0^t (K * u, \phi) d\xi. \quad (22)$$

Now, when we integrate (DP) and let $n \rightarrow +\infty$ ($\tau \rightarrow 0$) we obtain the following limit for the l.h.s.

$$\int_0^t (\partial_t u, \phi) d\xi + \int_0^t (\nabla u, \nabla \phi) d\xi + \int_0^t (g, \phi)_\Gamma d\xi + \int_0^t K(h, \phi) d\xi + \int_0^t (K * u, \phi) d\xi.$$

First, by (20) we have

$$\lim_{n \rightarrow +\infty} \int_0^t \|f(\bar{u}_n(\xi), \nabla \bar{u}_n(\xi)) - f(u(\xi), \nabla u(\xi))\| d\xi = 0.$$

Secondly, as f is Lipschitz we have

$$\begin{aligned} & \|f(\bar{u}_n(\xi - \tau), \nabla \bar{u}_n(\xi - \tau)) - f(\bar{u}_n(\xi), \nabla \bar{u}_n(\xi))\| \\ & \leq C \sqrt{\|\bar{u}_n(\xi - \tau) - \bar{u}_n(\xi)\|^2 + \|\nabla \bar{u}_n(\xi - \tau) - \nabla \bar{u}_n(\xi)\|^2} = C\tau \sqrt{\|\partial_t u_n(\xi)\|^2 + \|\nabla \partial_t u_n(\xi)\|^2} = C\tau \|\partial_t u_n(\xi)\|_{H^1(\Omega)} \end{aligned}$$

from which it follows that

$$\lim_{n \rightarrow +\infty} \left| \int_0^t (f(\bar{u}_n(\xi - \tau), \nabla \bar{u}_n(\xi - \tau)) - f(u(\xi), \nabla u(\xi)), \phi) d\xi \right|^2 \leq \|\phi\|^2 C \lim_{n \rightarrow +\infty} \tau \int_0^t \|\partial_t u_n(\xi)\|_{H^1(\Omega)}^2 d\xi = 0$$

as from Propositions 3.3 and 3.4 we have

$$\int_0^T \|\partial_t u_n(\xi)\|^2 d\xi + \int_0^T \|\nabla \partial_t u_n(\xi)\|^2 d\xi \leq C.$$

From the above we conclude that for the integrated r.h.s. of (DP) it holds

$$\lim_{n \rightarrow +\infty} \int_0^t (f(\bar{u}_n(\xi - \tau), \nabla \bar{u}_n(\xi - \tau)), \phi) d\xi = \int_0^t (f(u(\xi), \nabla u(\xi)), \phi) d\xi.$$

We conclude that taking the limit for $n \rightarrow +\infty$ ($\tau \rightarrow 0$) in (DP) results in

$$\int_0^t (\partial_t u, \phi) d\xi + \int_0^t (\nabla u, \nabla \phi) d\xi + \int_0^t (g, \phi)_\Gamma d\xi + \int_0^t K(h, \phi) d\xi + \int_0^t (K * u, \phi) d\xi = \int_0^t (f(u(\xi), \nabla u(\xi)), \phi) d\xi.$$

Taking the derivative with respect to t we arrive at (P).

Finally, we have to pass to the limit for $\tau \rightarrow 0$ in (DMPi) to arrive at (MP). This follows the same line as passing the limit in (DPi), therefore we skip the details. \square

The convergences of Rothe's functions towards the weak solution (P)-(MP) (as stated in the proof of Theorem 2) have been shown for a subsequence. Note, that taking into account Theorem 1 we see that the whole Rothe's functions converge against the solution.

Conclusion

A semilinear parabolic integro-differential problem of second order with an unknown convolution kernel is considered. The existence and uniqueness of a weak solution for the IBVP is proved. The missing integral kernel is recovered from an integral-type measurement. A numerical algorithm based on Rothe's method is established and the convergence of approximations towards the exact solution is demonstrated.

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References

- [1] J. Hadamard, *Lectures on Cauchy's Problem in Linear Partial Differential Equations*, New York: Dover, 1953.
- [2] H. W. Engl, M. Hanke, A. Neubauer, *Regularization of inverse problems*, Vol. 375 of *Mathematics and its Applications*, Kluwer Academic Publishers, Dordrecht, 1996.
- [3] A. Rieder, *No problems with inverse problems. An introduction to its stable solutions.*, Vieweg, Wiesbaden, 2003.
- [4] V. Isakov, *Inverse Problems for Partial Differential Equations*, *Applied Mathematical Sciences*, Springer, New York, 2006.
- [5] R. Miller, *Optimization: foundations and applications*, Wiley-Interscience, New York, 2000.
- [6] P. Pedregal, *Introduction to optimization*, Springer, New York, 2003.
- [7] A. Kufner, O. John, S. Fučík, *Function Spaces, Monographs and textbooks on mechanics of solids and fluids*, Noordhoff International Publishing, Leyden, 1977.
- [8] A. Prilepko, D. Orlovsky, I. Vasin, *Methods for solving inverse problems in mathematical physics.*, *Pure and Applied Mathematics*, Marcel Dekker. 231. New York, NY: Marcel Dekker, 2000.
- [9] M. Ismailov, F. Kanca, D. Lesnic, *Determination of a time-dependent heat source under nonlocal boundary and integral overdetermination conditions.*, *Appl. Math. Comput.* 218 (8) (2011) 4138–4146.
- [10] M. Slodička, *Recovery of boundary conditions in heat transfer*, in: O. Fudym, J.-L. Battaglia, G. D. et al. (Eds.), *IPDO 2013 : 4th Inverse problems, design and optimization symposium*, 2013 June 26–28, Albi, Ecole des Mines d'Albi-Carmaux, 10p., 2013, ISBN 979-10-91526-01-2.
- [11] M. Renardy, W. J. Hrusa, J. A. Nohel, *Mathematical problems in viscoelasticity.*, *Pitman Monographs and Surveys in Pure and Applied Mathematics*, 35. Harlow: Longman Scientific & Technical; New York: John Wiley & Sons, Inc., 1987.
- [12] J. W. Delleur, *The Handbook of Groundwater Engineering*, Springer CRC Press, 1999.
- [13] F. Colombo, D. Guidetti, A. Lorenzi, *On applications of maximal regularity to inverse problems for integrodifferential equations of parabolic type.*, Ruiz Goldstein, Gisèle (ed.) et al., *Evolution equations. Proceedings of the conference*, Blaubeuren, Germany, June 11–17, 2001 in honor of the 60th birthdays of Philippe Bérnilan, Jerome A. Goldstein and Rainer Nagel. New York, NY: Marcel Dekker. *Lect. Notes Pure Appl. Math.* 234, 77–89 (2003). (2003).
- [14] F. Colombo, D. Guidetti, V. Vespi, *Some global in time results for integrodifferential parabolic inverse problems.*, Favini, Angelo (ed.) et al., *Differential equations. Inverse and direct problems. Papers of the meeting*, Cortona, Italy, June 21–25, 2004. Boca Raton, FL: CRC Press. *Lecture Notes in Pure and Applied Mathematics* 251, 35–58 (2006). (2006).
- [15] F. Colombo, D. Guidetti, *A global in time existence and uniqueness result for a semilinear integrodifferential parabolic inverse problem in Sobolev spaces.*, *Math. Models Methods Appl. Sci.* 17 (4) (2007) 537–565.
- [16] D. Guidetti, *Convergence to a stationary state for solutions to parabolic inverse problems of reconstruction of convolution kernels.*, *Differ. Integral Equ.* 20 (9) (2007) 961–990.
- [17] F. Colombo, D. Guidetti, *Some results on the identification of memory kernels*, in: M. Ruzhansky, J. Wirth (Eds.), *Modern aspects of the theory of partial differential equations. Including mainly selected papers based on the presentations at the 7th international ISAAC congress*, London, UK, July 13–18, 2009., *Operator Theory: Advances and Applications* 216. Basel: Birkhäuser, 2011, pp. 121–138.
- [18] K. Rektorys, *The method of discretization in time and partial differential equations*. Transl. from the Czech by the author., *Mathematics and Its Applications (East European Series)*, Vol. 4. Dordrecht - Boston - London: D. Reidel Publishing Company; Prague: SNTL - Publishers of Technical Literature, 1982.
- [19] J. Kačur, *Method of Rothe in evolution equations*, Teubner-Texte zur Mathematik, 1985.
- [20] H. Gajewski, K. Gröger, K. Zacharias, *Nichtlineare Operatorgleichungen und Operatordifferentialgleichungen.*, *Mathematische Lehrbücher und Monographien. II. Abteilung. Band 38*. Berlin: Akademie-Verlag, 1974.
- [21] D. Bainov, P. Simeonov, *Integral inequalities and applications*, *Mathematics and Its Applications. East European Series.* 57. Dordrecht, Kluwer Academic Publishers, 1992.
- [22] L. A. Ljusternik, V. I. Sobolev, *Elements of functional analysis*, Nauka, Moscow, 1965, Russian.
- [23] J. Nečas, *Direct Methods in the Theory of Elliptic Equations*, *Springer Monographs in Mathematics*, 2012.
- [24] W. Rudin, *Real and complex analysis*, McGraw-Hill, 1987.